Robust and Efficient Degeneracy Detection for Curved Solids

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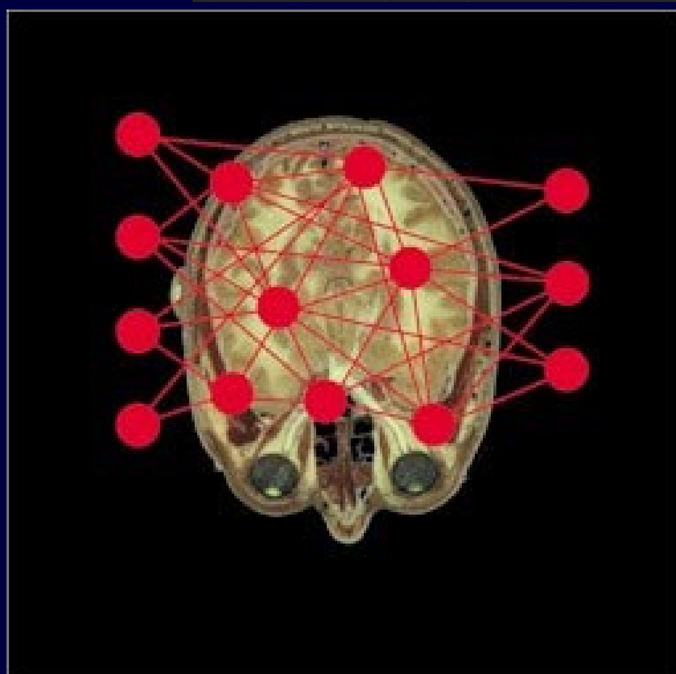
OUTLINE

- Applications of Polynomial System Solving
- **Toric** Techniques (Resultants)
- **Alpha Theory**: "Exact" Computation via Smart Approximation
- Focus on Solid Modelling

APPLICATIONS



CONCRETE EXAMPLE

Consider the following **family** of

6×6 system involving **bivariate bicubic**

polynomials $X_i^{\pm}, Y_i^{\pm}, Z_i^{\pm}$: ($\bigcap$ 3 bicubic patches)

$$
\begin{aligned}
X_1^+(s_1,t_1)X_2^-(s_2,t_2) &= X_2^+(s_2,t_2)X_1^-(s_1,t_1)\\
Y_1^+(s_1,t_1)Y_2^-(s_2,t_2) &= Y_2^+(s_2,t_2)Y_1^-(s_1,t_1)\\
Z_1^+(s_1,t_1)Z_2^-(s_2,t_2) &= Z_2^+(s_2,t_2)Z_1^-(s_1,t_1)\\
X_2^+(s_2,t_2)X_3^-(s_3,t_3) &= X_3^+(s_3,t_3)X_2^-(s_2,t_2)\\
Y_2^+(s_2,t_2)Y_3^-(s_3,t_3) &= Y_3^+(s_3,t_3)Y_2^-(s_2,t_2)\\
Z_2^+(s_2,t_2)Z_3^-(s_3,t_3) &= Z_3^+(s_3,t_3)Z_2^-(s_2,t_2)
\end{aligned}
$$

HOW BIG ARE THE COMPUTATIONS?

•Bicubic Patches + Offsets + Boolean Ops $\Longrightarrow$ intersecting curves of degree in the **100's**, specified by polynomials with number of digits in the **100's** $\Longrightarrow$ linear algebra on matrices of size ≈ 10000 and big entries...

•Instead of reducing algebraically, why not **work directly with the higher dimensional problem?** Recent toric advances enable us to do so!

EXACT VS. FLOATING POINT

- Floating point arithmetic is fast but can easily lead to erroneous results...
- Exact arithmetic can certify one's answers but is too slow...
- Bit model poorly suited for solid modelling...
- Enter numerical conditioning $(\kappa, \mu, \ldots)$ and round-off considerations...

SMART APPROXIMATE COMPUTATION

• Modern approach to solving $F(x) = 0$ is to use algorithms with complexity explicitly bounded in terms of some sort of **degree** and **condition number** for F

• One then attempts to get explicit **probabilistic** bounds on the underlying condition number $\Longrightarrow$ **rigourous and realistic** probabilistic complexity bounds.

MODERN NUMERICAL AG #1

Theorem 1 *[Shub-Smale, 1994]* *Consider $n \times n$ polynomial systems $F := (f_1, \ldots, f_n)$ where $n > 1$ and $\deg f_i = d_i$. Then there is a natural probability measure on these F such that $\mathrm{Prob}(\mu_{\mathrm{norm}}(F) > C)$ is no more than*

$$\frac{n^3(n+1)\left(\sum_i \binom{d_i+n}{n}\right)^2 \prod_i d_i}{C^4}.$$

MODERN NUMERICAL AG #2

Theorem 2 *[Malajovich-Rojas, 2001] Consider unmixed systems F with any fixed Newton polytope P. Then there is a natural metric on these F (and the incidence variety $\{(F, \zeta)\}$) such that* $\mathrm{dist}((F, \zeta), \mathrm{Singular}) = \frac{1}{\kappa(F,\zeta)}$.

i.e., **Big** $\kappa(F, \zeta) \Longrightarrow$ roots are getting **dangerously close**.

MODERN NUMERICAL AG #3

Theorem 3 *[Rojas, 1999+]* *You can ε-approximate a set of points intersecting every connected component of the complex zero set of* **any** *F in $\{\|z\| < R\}$, within*

$$O\left(n^4 D^3 M_F^{3.376} \log\log\frac{R}{\varepsilon}\right)$$

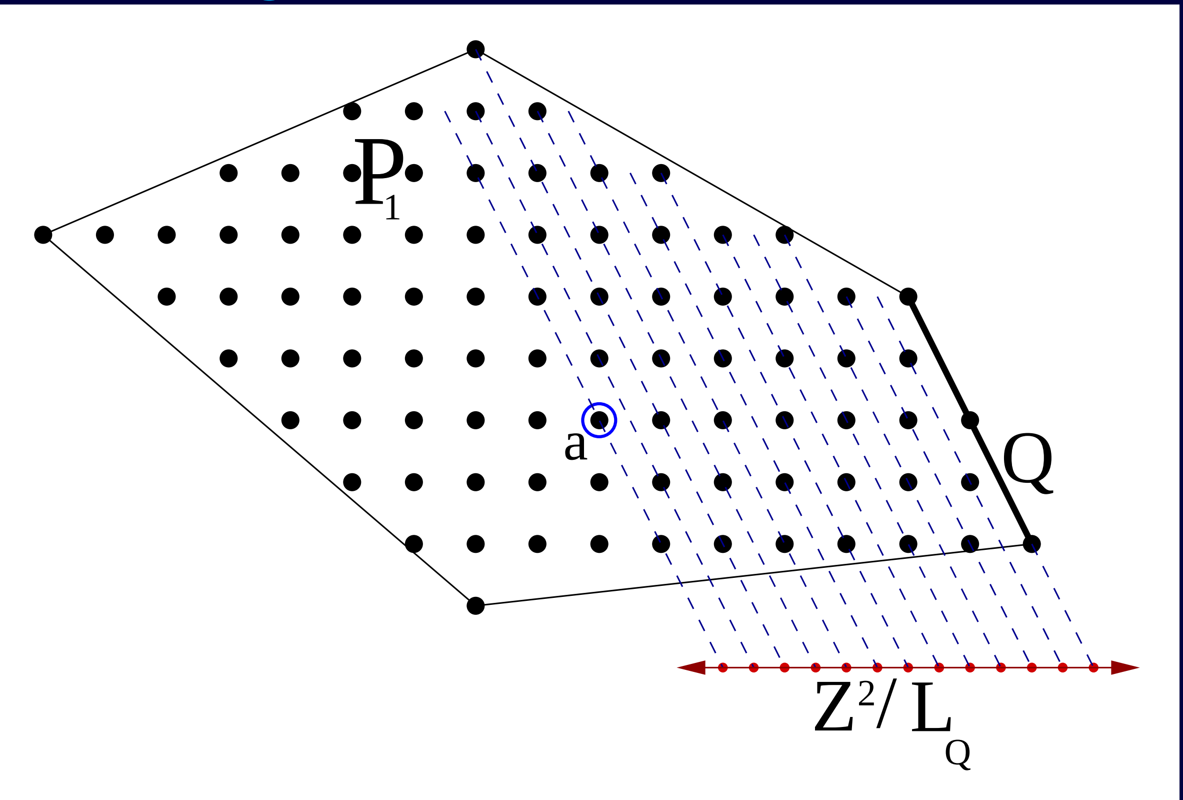
arithmetic operations, and a similar bound replacing $\log\frac{R}{\varepsilon}$ by $\kappa(F)$ holds.

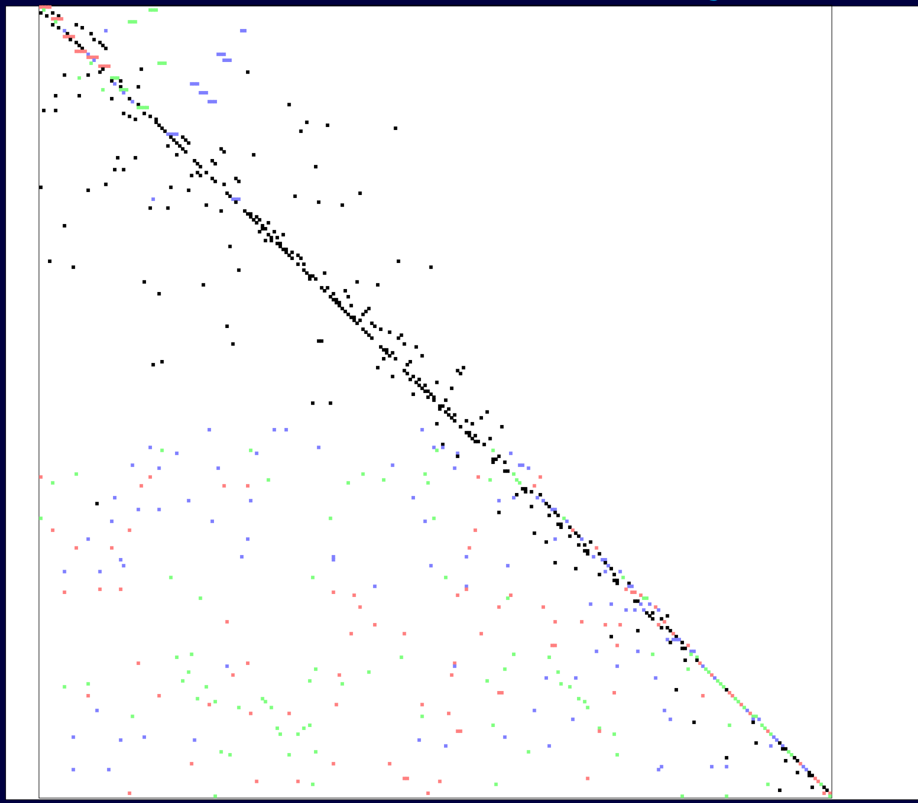
...M_F=Matrix Size

TORIC RESULTANTS #1...

Input polynomials $\longrightarrow$ $\boxed{\text{Supports}}$ & Coefficients $\longrightarrow$

Combinatorics... $\longrightarrow$ Matrix Encodes Roots



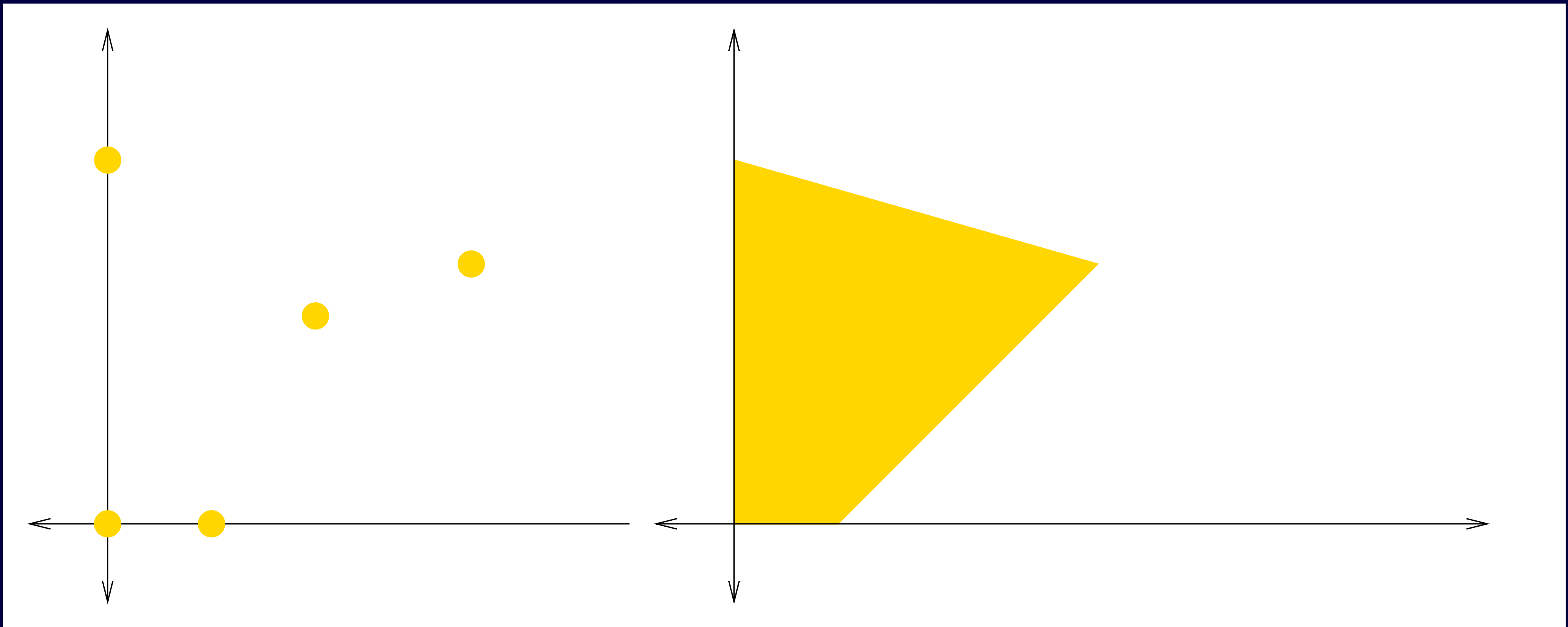


[Gelf'and-Kapranov-Zelevinsky, 1991; Canny-Emiris & Sturmfels, 1993; Mourrain-Pan, 1997; Rojas, 1999; Goldman-Zhang, 1999; Dickenstein-D'Andrea, 2000; Hong-Minimair, 2000; Khetan, 2002]

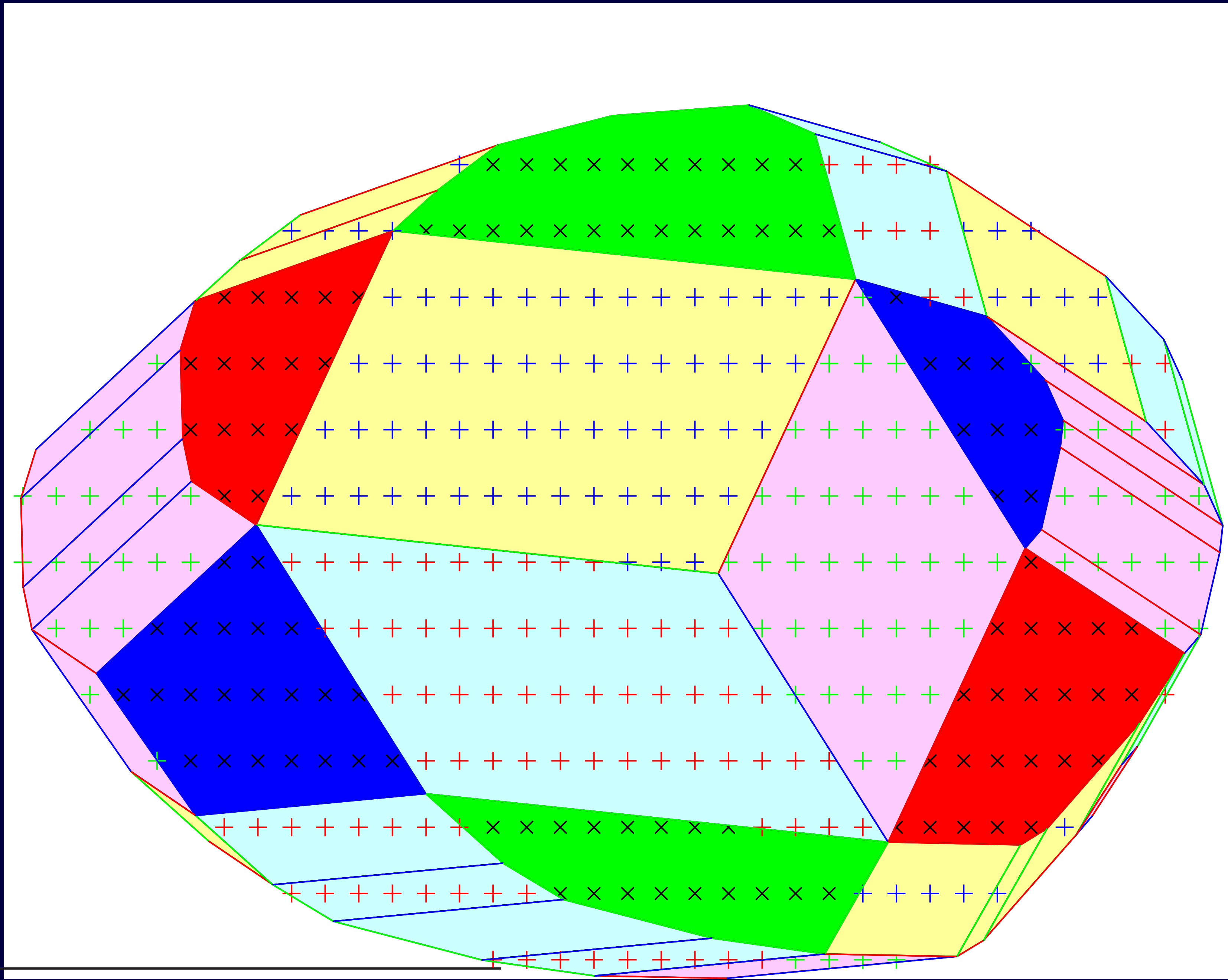
NEWTON POLYTOPES...

$$\mathrm{Newt}(7 - 2.5x_1^2 + 27x_1^7x_2^5 + \sqrt{7}x_1^4x_2^4 + ex_2^7) =$$

$$\mathrm{Newt}(7x_1^0x_2^0 - 2.5x_1^2x_2^0 + 27x_1^7x_2^5 + \sqrt{7}x_1^4x_2^4 + ex_1^0x_2^7)$$



TORIC RESULTANTS #2...



MODERN NUMERICAL AG #4

Theorem 4 *[Cucker-Smale, 1999]* κ *and* μ_{norm} *(along with dimension and number of monomial terms) explicitly determine how much precision is needed in essentially* **any** *semi-algebraic feasibility problem!*

[Keyser, Malajovich, Rojas, 2002] Sharpening bounds in context of polynomial system solving over $\mathbb{C}$ and $\mathbb{R}$...
[Dunagan, Spielman, Teng, 2002] LP...

COMPARING METHODS #1

Toric Resultants:

- Can reduce any geometric question (over $\mathbb{R}$ or $\mathbb{C}$) to essentially **(a)** structured eigenproblem and/or **(b)** determinants and interpolation
- Easily parallelizable
- High precision on well-conditioned inputs with high probability...

[Jónsson-Vavasis (Dense 2×1 and 3×2), 2001]
[Rojas (Toric 4×3), 2002]

COMPARING METHODS #2

Gröbner Bases

•Hard to parallelize

•Coefficient swell and degree explosion still not well understood

•There are systems with $O(n)$ polynomials of degree ≤ 5 in $O(n)$ variables where **any** Gröbner basis has a generator of degree 2^{2^n}.

COMPARING METHODS #3

Homotopy Methods

- Can be slow when starting from scratch...
- ...but polyhedral version can be combined nicely with toric resultants after some pre-processing, yielding a super-fast hybrid method.

EFFICIENT REAL METHODS?


Askold G. Khovanski's Theorem on Fewnomials (1980): Number of **non-degenerate** roots in $\mathbb{R}_+^n$ is $\leq (n+1)^m 2^{m(m-1)/2}$, where $m =$ total number of monomial term.

e.g., 2×2 **trinomial system** $\Longrightarrow m = 5 \Longrightarrow$ ≤ 248832 roots in the positive quadrant.

Li-Rojas-Wang:

The correct bound is actually 5 (!)

♡ Thank you for listening!

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- See...

  `www.math.tamu.edu/~rojas`

  for on-line papers and further information...

  `www.math.tamu.edu/conferences/cbms`

  for a highly relevant CBMS lecture series going on right now! And...

  `www.maf.vu.lt/katedros/cs2/cagl/aggm`

  for an upcoming conference in Vilnius on Algebraic Geometry and Geometric Modelling...